الجامعة السورية الخاصة
SYRIAN PRIVATE UNIVERSITY

كـلـيــة هندسة الحاسوب والمعلوماتية
Computer and Informatics Engineering Faculty

Electric Circuits I

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Chapter 7
First-Order Circuits

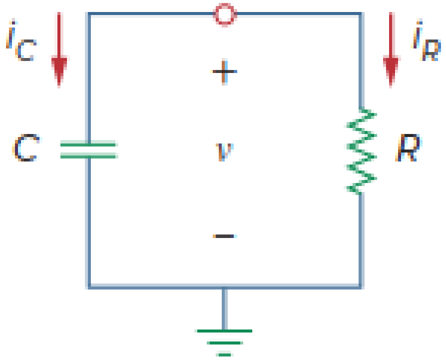
Introduction

- A simple circuit comprising a resistor and capacitor is called ***RC* circuit**;
- A simple circuit comprising a resistor and an inductor is called ***RL* circuit**.
- These types of circuits find continual applications in electronics, communications, and control systems.
- We carry out the analysis of *RC* and *RL* circuits by applying Kirchhoff's laws:
  - Apply Kirchhoff's laws to purely resistive circuits results in algebraic equations;
  - Apply the laws to *RC* and *RL* circuits produces differential equations.
  - The differential equations resulting from analyzing *RC* and *RL* circuits are of the first order.
  - Hence, the circuits are collectively known as **first-order circuits**.

✓ **A first-order circuit is characterized by a first-order differential equation.**

7.1 The Source-Free *RC* Circuit

- A **source-free *RC* circuit** occurs when its dc source is suddenly disconnected.
  - The energy already stored in the capacitor is released to the resistors.



- Applying KCL at the top node gives:

$$i_C + i_R = 0 \Leftrightarrow C\frac{dv}{dt} + \frac{v}{R} = 0 \quad \text{or} \quad \frac{dv}{dt} + \frac{v}{RC} = 0$$

  - This is a first-order differential equation.

The natural response of *RC* Circuit

- Since the capacitor is *initially charged*, we can assume that at time $t = 0$, the *initial voltage* is $v(0) = V_0$, with the corresponding value of the *energy stored* as

$$w(0) = \frac{1}{2}CV_0^2$$

- The first-order differential equation can be rewrite as

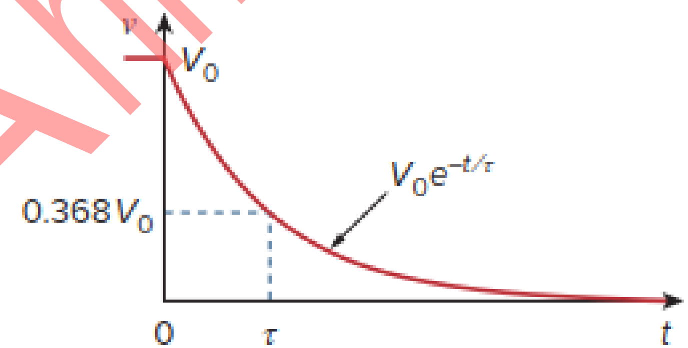
$$\frac{dv}{dt} + \frac{v}{RC} = 0 \Leftrightarrow \frac{dv}{v} = -\frac{1}{RC}dt$$

  - Integrating both sides, we get $\ln v = -\frac{t}{RC} + \ln A \Leftrightarrow \ln\frac{v}{A} = -\frac{t}{RC}$

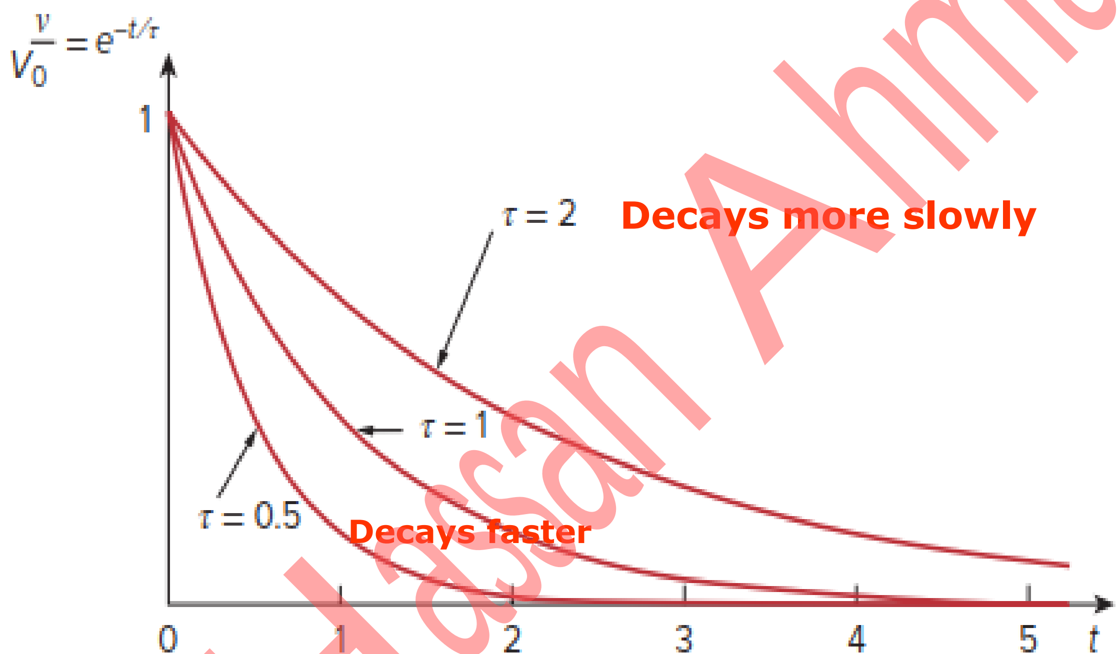
    where $\ln A$ is the integration constant.
  - Taking powers of *e* produces: $v(t) = Ae^{-t/RC}$
  - From the *initial conditions* $v(0) = A = V_0$. Hence, $\boxed{v(t) = V_0 e^{-t/RC}}$
  - This Eq. is called the **natural response of the circuit**.

- The *natural response* of a circuit refers to the behavior (in terms of voltages and currents) of the circuit itself, with no external sources of excitation.
- The natural response is illustrated graphically in Fig. with time constant $\tau = RC$.
- The time constant τ of a circuit is the *time* required for the response to decay to a factor of $1/e$ or 36.8 percent of its initial value.
- At $t = \tau$ ➔ $V_0 e^{-t/RC} = V_0 e^{-1} = 0.368 V_0$
- In terms of the *time constant*, Eq. for The *natural response* can be written as

$$v(t) = V_0 e^{-t/\tau} \quad or \quad \frac{v(t)}{V_0} = e^{-t/\tau}$$



- *Natural response* as $\frac{v(t)}{V_0} = e^{-t/\tau}$

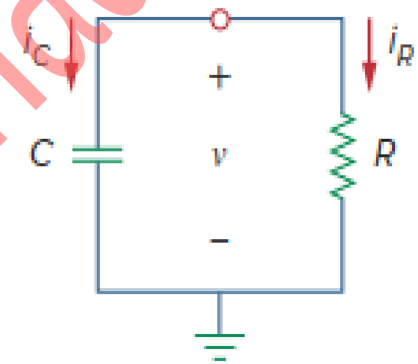


- v decays faster for small τ and slower for large τ.

Current, Power and Energy of *RC* Circuit

- The **current $i_R(t)$** is defined as

$$i_R(t)=\frac{v(t)}{R}=\frac{V_0}{R}e^{-t/\tau}$$



- The **power** dissipated in the resistor is

$$p(t)=vi_R=\frac{V_0^2}{R}e^{-2t/\tau}$$

- The **energy absorbed** by the resistor *up to time* t is

$$w_R(t)=\int_0^t p(\lambda)d\lambda=\int_0^t \frac{V_0^2}{R}e^{-2\lambda/\tau}d\lambda=-\frac{\tau V_0^2}{2R}e^{-2\lambda/\tau}\Big|_0^t=\frac{1}{2}CV_0^2\left(1-e^{-2t/\tau}\right),\quad \tau=RC$$

- The **initial energy stored** in the capacitor is

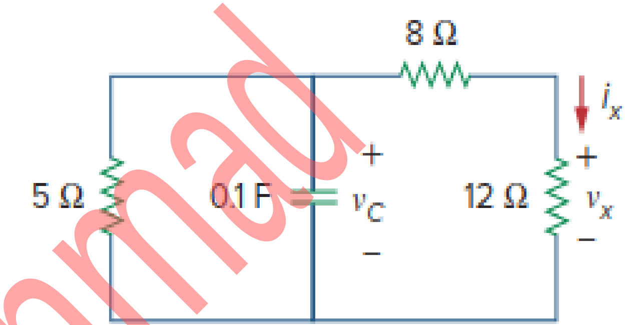
$$w_C(0)=\frac{1}{2}CV_0^2$$

The Key to Working with a Source-Free RC Circuit Is Finding:

1. The initial voltage $v(0)=V_0$ across the capacitor.
2. The time constant τ.

Example 7.1

In Fig., let $v_C(0) = 15$ V. Find v_C, v_x, and i_x for $t > 0$.



Solution

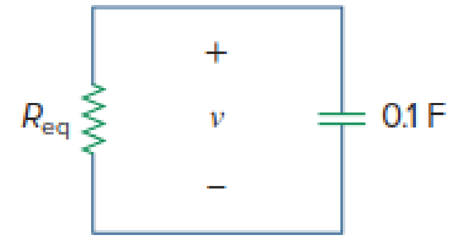
First, conform the given circuit to the standard RC circuit.

The 8- Ω and 12- Ω *resistors* in series = 20 Ω.

This 20-Ω resistor in *parallel* with the 5-Ω resistor ➔

$$R_{eq} = \frac{20\times 5}{20+5} = 4\Omega$$

Hence, the equivalent circuit is as shown in Fig.



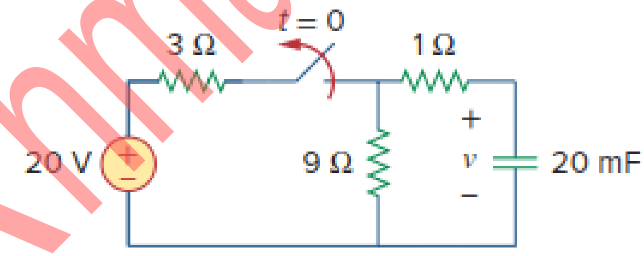
The time constant is $\tau = R_{eq}C = 4(0.1) = 0.4\text{s}$

Thus, $v = v(0)e^{-t/\tau} = 15e^{-t/0.4}\ \text{V} \Rightarrow v_C = v = 15e^{-2.5t}\ \text{V}$

Use VDR to get: $v_x = \frac{12}{12+8}v = 0.6\left(15e^{-2.5t}\right) = 9e^{-2.5t}\ \text{V}$

Finally, $i_x = \frac{v_x}{12} = 0.75e^{-2.5t}\ \text{A}$

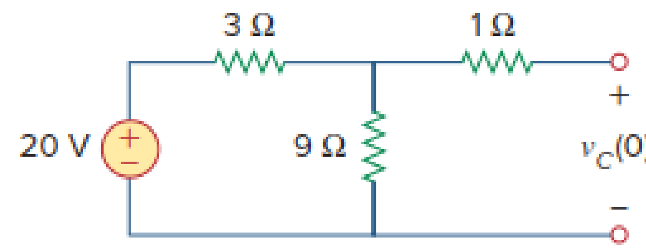
Example 7.2. The switch in the circuit in Fig. has been closed for a long time, and it is opened at $t = 0$. Find $v(t)$ for $t \geq 0$. Calculate the initial energy stored in the capacitor.



Solution

For $t < 0$, the switch is closed; the capacitor is an *open circuit to dc*, Fig.(a). Using VDR

$$v_C(t) = \frac{9}{9+3}(20) = 15V, \quad t < 0$$

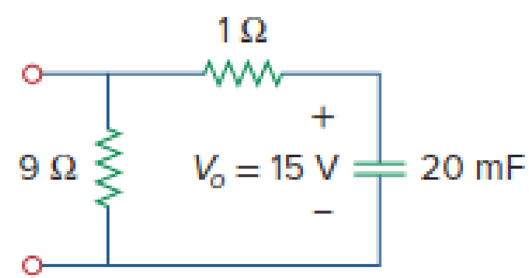


(a)

Since the voltage across a capacitor *cannot change instantaneously*, the voltage across the capacitor at $t = 0^-$ is the same at $t = 0$, or $v_C(0) = V_0 = 15\,V$

For $t > 0$, the switch is opened, and we have the RC circuit shown in Fig. (b).

$$R_{eq} = 1 + 9 = 10\Omega \Rightarrow \tau = R_{eq}C = 10 \times 20 \times 10^{-3} = 0.2\text{s}$$



(b)

Thus, the voltage across the capacitor for $t \geq 0$ is

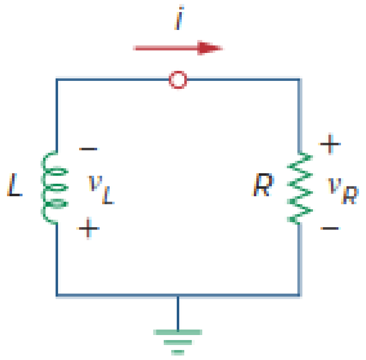
$$v(\text{t}) = v(0)e^{-t/\tau} = 15e^{-t/0.2}\text{ V} \quad \text{or} \quad v(\text{t}) = 15e^{-5t}\text{ V}$$

The initial energy stored in the capacitor is

$$w_C(0) = \frac{1}{2}CV_0^2(0) = \frac{1}{2} \times 20 \times 10^{-3} \times (15)^2 = 2.25\text{ J}$$

7.2 The Source-Free *RL* Circuit

- A **first-order RL circuit** consists of a inductor *L* (or its equivalent) and a resistor (or its equivalent).
  - The circuit response is assumed to be the current $i(t)$ through the inductor.
  - At $t = 0$, we assume that the inductor has an **initial current** I_0, or $$i(0) = I_0$$



with the corresponding **energy** stored in the *inductor* as

$$w(0) = \frac{1}{2} L I_0^2$$

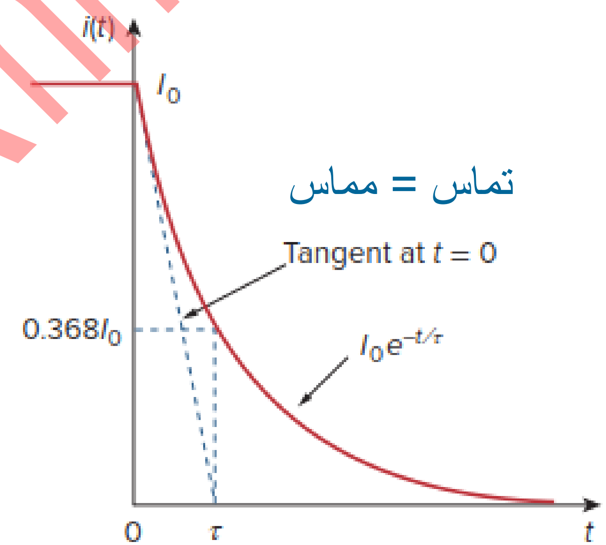
  - Applying KVL around the loop in Fig.

$$v_L + v_R = 0; \quad v_L = L\, di/dt, \quad v_R = iR$$

$$\Rightarrow L\frac{di}{dt} + Ri = 0 \Rightarrow \frac{di}{i} = -\frac{R}{L}dt \Rightarrow \int_{I_0}^{i(t)} \frac{di}{i} = -\int_0^t \frac{R}{L} dt$$

$$\Rightarrow i(t) = I_0 e^{-Rt/L}$$

- The **time constant** for the *RL* circuit is $\tau = \dfrac{L}{R}$ (second)
- The **current** through the inductor is, $i(t) = I_0 e^{-t/\tau}$
- The **current response** of the RL circuit is as in Fig.



- The **voltage** *across the resistor* is

$$v_R(t) = iR = I_0 R e^{-t/\tau}$$

- The **power** *dissipated* in the resistor is

$$p = v_R i = I_0^2 R e^{-2t/\tau}$$

- The energy absorbed by the resistor is

$$w_R(t) = \frac{1}{2} L I_0^2 \left(1 - e^{-2t/\tau}\right), \quad \tau = L/R$$

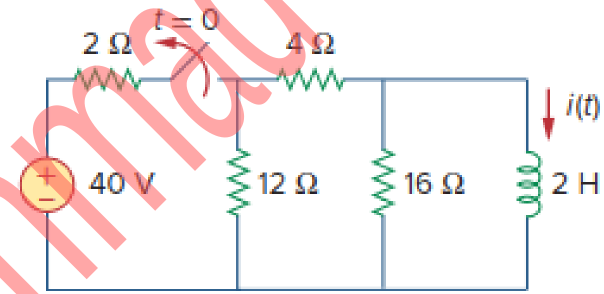
- Note that as $t \to \infty,\ w_R(\infty) \to \frac{1}{2} L I_0^2$, which is the same as $w_L(0)$, the **initial energy stored** in the inductor.

The Key to Working with a Source-Free RL Circuit is to Find:

The initial current $i(0) = I_0$ through the inductor.

The time constant τ of the circuit.

Example 7.3. The switch in the circuit in Fig. has been closed for a long time, and it is opened at $t = 0$. Find $i(t)$ for $t > 0$.



Solution

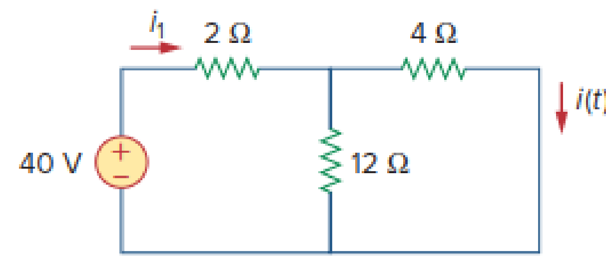
For $t < 0$, the switch is closed; and the inductor acts as a short circuit to dc. The 16-Ω resistor is short-circuited; the resulting circuit is shown in Fig.(a).

The 4- Ω and 12-Ω resistors in parallel: $4 \| 12 = 3\Omega$

The 2- Ω and 3-Ω resistors in series: 2 + 3= 5Ω .

Hence, $$i_1 = \frac{40}{5} = 8\,\text{A}$$

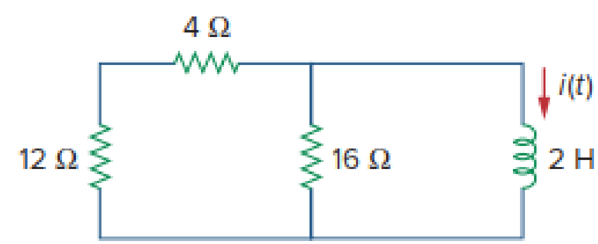
Using CDR, we get $$i(\text{t}) = \frac{12}{12+4} i_1 = 6\,\text{A}, \quad t < 0$$



(a)

Since the current through an inductor cannot change instantaneously, $$i(0) = i(0^-) = 6\,\text{A}$$

When $t > 0$, the switch is open and the voltage source is disconnected. We now have the source-free RL circuit in Fig. (b). $$R_{\text{eq}} = (12+4) \| 16 = 8\Omega$$

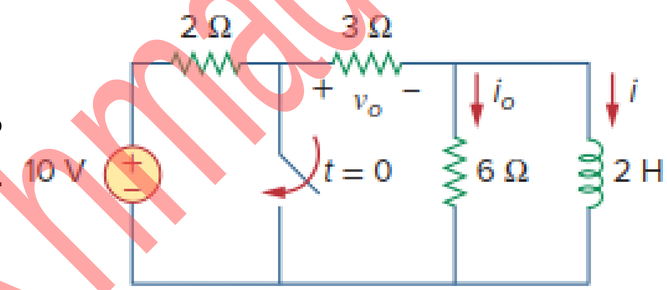


(b)

The time constant is $$\tau = \frac{L}{R_{\text{eq}}} = \frac{2}{8} = \frac{1}{4}\,\text{s}$$

Thus, $$i(t) = i(0)e^{-t/\tau} = 6e^{-4t} \text{ A}$$

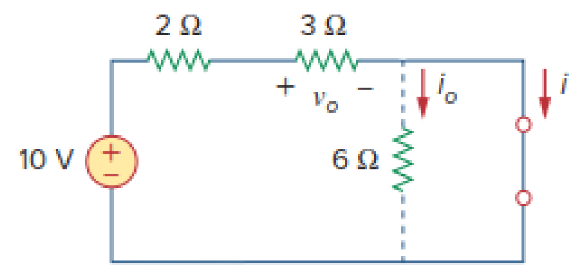
Example 7.4. In the circuit shown in Fig. , find i_o, v_o, and i for **all time**, assuming that the switch was open for a long time. Plot of i and i_o.



Solution

For $t < 0$, the switch is open. Since the inductor acts like a short circuit to dc, the 6-Ω resistor is short-circuited, Fig. (a). Hence, $i_o = 0$, and

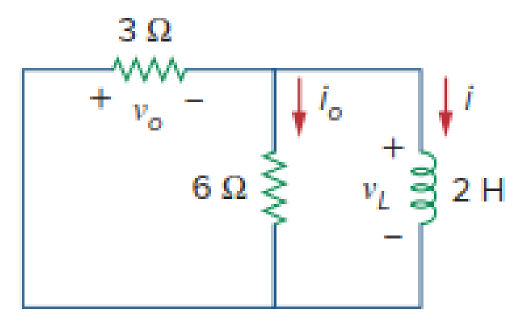
$$i(t) = \frac{10}{2+3} = 2\,\text{A},\ \ t<0 \Rightarrow v_o(t) = 3i(t) = 6\,\text{V},\ \ t<0 \Rightarrow i(0) = 2\,\text{A}$$



(a)

For $t > 0$, the switch is closed, so that the voltage source is short-circuited. We now have a source-free RL circuit as shown in Fig. (b).

At the inductor terminals: $R_{eq} = R_{Th} = 3 \| 6 = 2\Omega$



(b)

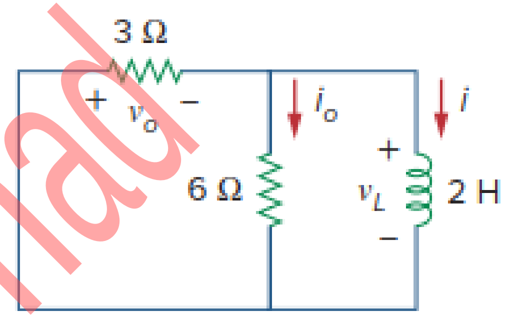
The time constant is

$$\tau = \frac{L}{R_{Th}} = \frac{2}{2} = 1\text{s} \Rightarrow i(t) = i(0)e^{-t/\tau} = 2e^{-t}\ \text{A}, \quad t > 0$$

Because the inductor is in parallel with the 6- and 3-Ω resistors,

$$v_o(t) = -v_L = -L\frac{di}{dt} = -2(-2e^{-t}) = 4e^{-t}\ \text{V},\ t>0$$

$$i_o(t) = \frac{v_L}{6} = \frac{-v_o}{6} = -\frac{4e^{-t}}{6} = -\frac{2}{3}e^{-t}\ \text{A},\quad t>0$$

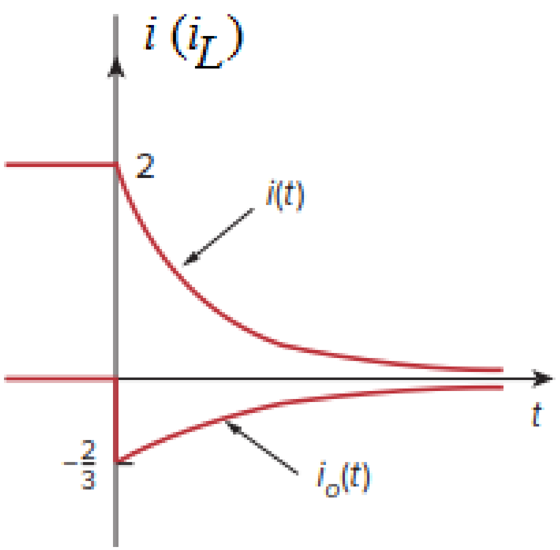


(b)

Thus, for all time,

$$i_o(t) = \begin{cases} 0\,\text{A}, & t<0 \\ -\frac{2}{3}e^{-t}\ \text{A}, & t>0 \end{cases},\qquad v_o(t) = \begin{cases} 6\ \text{V}, & t<0 \\ 4e^{-t}\ \text{V}, & t>0 \end{cases}$$

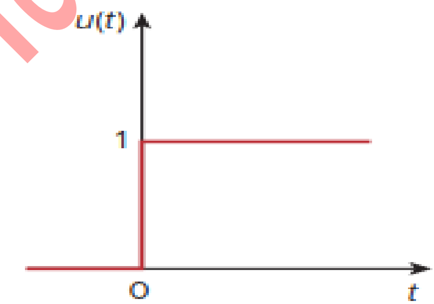
$$i(t) = \begin{cases} 2\,\text{A}, & t<0 \\ 2e^{-t}\ \text{A}, & t\ge 0 \end{cases}$$



7.3 Unit-Step Function

- The **unit step function *u*(*t*)** is 0 for negative values of t and 1 for positive values of t.

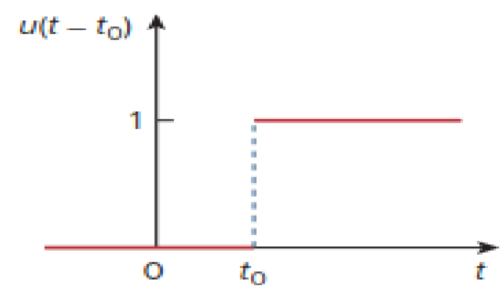
$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$



- At $t = 0$, where it changes abruptly from 0 to 1.

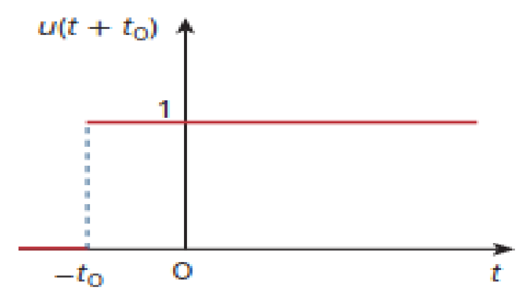
- At $t = t_0$ (where $t_0 > 0$):

$$u(t - t_0) = \begin{cases} 0, & t < t_0 \\ 1, & t > t_0 \end{cases} \qquad (u(t) \text{ is delayed by } t_0 \text{ seconds})$$



- At $t = -t_0$

$$u(t + t_0) = \begin{cases} 0, & t < -t_0 \\ 1, & t > -t_0 \end{cases} \qquad (u(t) \text{ is advanced by } t_0 \text{ seconds})$$



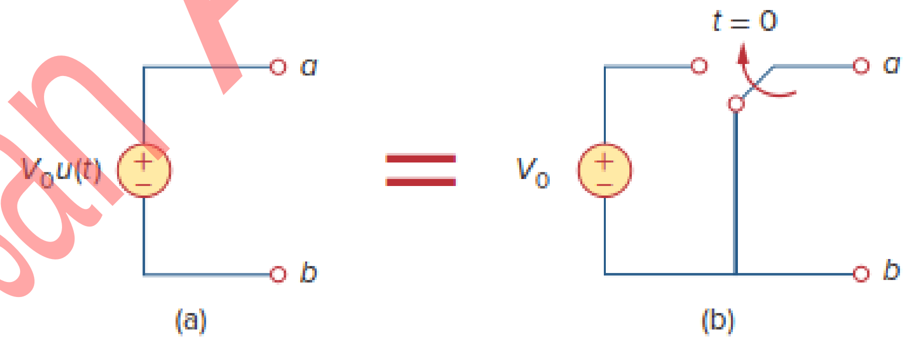
- The **step function** is used to represent an **abrupt change** in **voltage** or **current**, like the changes that occur in the circuits of control systems and digital computers.
- **For example**, the voltage

$$v(t) = \begin{cases} 0, & t < t_0 \\ V_0, & t > t_0 \end{cases}$$

may be expressed in terms of the **unit step function** as $v(t) = V_0 u(t - t_0)$

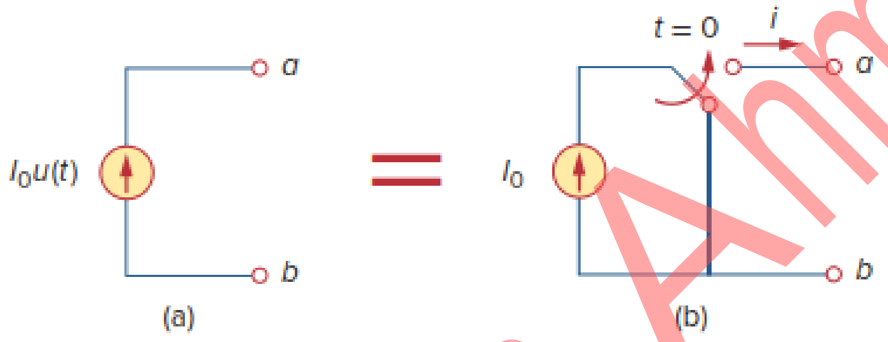
- at $t_0 = 0$, then $v(t)$ is simply the step voltage $V_0 u(t)$.

- (a) Voltage source of $\boldsymbol{V_0 u(t)}$,
- (b) its equivalent circuit.



- For $t < 0$ ➔ the terminals a-b are short-circuited ($v = 0$),
- For $t > 0$ ➔ at the terminals a-b appear $v = V_0$,

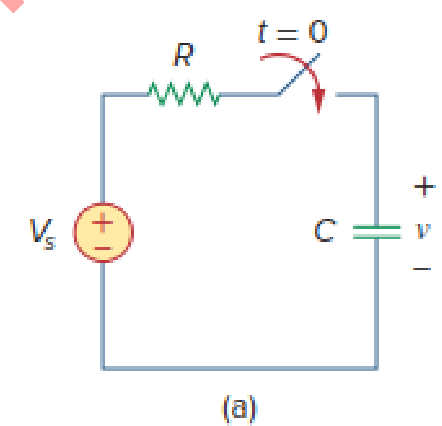
- Similarly, a **current source** of $I_0u(t)$ is shown in Fig. (a), while its *equivalent circuit* is in Fig. (b).



- For $t < 0$ ➔ there is an open circuit ($i = 0$),
- For $t > 0$ ➔ flow current $i = I_0$,

7.4 The Step-Response of a RC Circuit

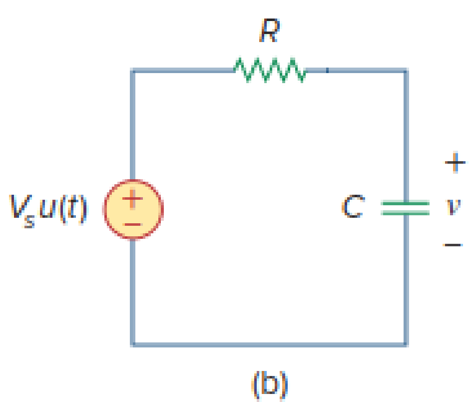
- The **step response** of a circuit is its behavior when the excitation (إثارة = تحريض) is the step function, which may be a voltage or a current source.
  - **Initial condition**: $v(0^-) = v(0^+) = V_0$
    where $v(0^-)$ is the voltage across the capacitor *just before switching*, $v(0^+)$ is its voltage *immediately* after switching.
  - Applying KCL:



$$C\frac{dv}{dt} + \frac{v - V_s u(t)}{R} = 0 \Leftrightarrow \frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC}u(t)$$

For $t > 0$ ➔

$$\frac{dv}{dt} + \frac{v}{RC} = \frac{V_s}{RC} \Rightarrow \frac{dv}{dt} = -\frac{v - V_s}{RC} \text{ or } \frac{dv}{v - V_s} = -\frac{dt}{RC}$$



By integrating and taking the exponential, we have

$$v(t) = V_s + (V_0 - V_s)e^{-t/\tau}, \quad \tau = RC, \quad t > 0$$

- The **complete response (or total response)** of a ***RC* circuit** to a *sudden application* of a *dc voltage* source with *initially charged capacitor* (with V_0).

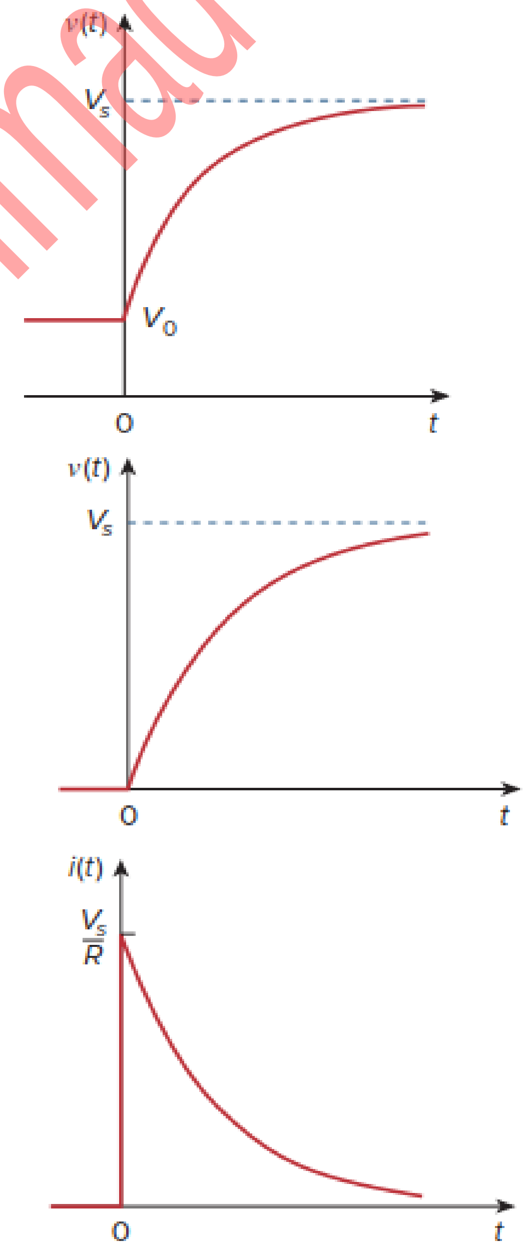
$$v(t)=\begin{cases} V_0, & t<0 \\ V_s+(V_0-V_s)e^{-t/\tau}, & t>0 \end{cases}$$

- If the capacitor is *uncharged* initially ($V_0 = 0$).

$$v(t)=\begin{cases} 0, & t<0 \\ V_s(1-e^{-t/\tau}), & t>0 \end{cases} \Leftrightarrow v(t)=V_s(1-e^{-t/\tau})u(t)$$

$$\Rightarrow i(t)=C\frac{dv}{dt}=\frac{C}{\tau}V_s e^{-t/\tau}, \quad \tau=RC, \; t>0$$

$$\Rightarrow i(t)=\frac{V_s}{R}e^{-t/\tau}u(t)$$



- The **complete response can be looked by tow ways:**
- **One way:**

$$\text{Complete response} = \underbrace{\text{natural response}}_{\text{stored energy}}(v_n) + \underbrace{\text{forced response}}_{\text{independent source}}(v_f)$$

$$v = v_n + v_f; \quad v_n = V_0 e^{-t/\tau}, \quad v_f = V_s(1 - e^{-t/\tau})$$

- **Another way:**

$$\text{Complete response} = \underbrace{\text{transient response}}_{\text{temperary part}}(v_t) + \underbrace{\text{steady-state response}}_{\text{permanent part}}(v_{ss})$$

$$\boxed{v = v_t + v_{ss}; \quad v_t = (V_0 - V_s)e^{-t/\tau}, \quad v_{ss} = V_s}$$

- The **transient response** is the circuit's temporary response (الاستجابة المؤقتة) that will die out (تنقرض = تموت) with time.
- The **steady-state response** is the behavior of the circuit a *long time after* an external excitation is applied.
- Finally, the **complete response can be written as:**

$$\boxed{v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau}}$$

- where $v(0)$ is the *initial voltage* at $t = 0^+$ and $v(\infty)$ is the *final or steady-state value.*

Three steps to find out the *step response* of an *RC* circuit:

1. The initial capacitor voltage $v(0)$.
2. The final capacitor voltage $v(\infty)$.
3. The time constant τ.

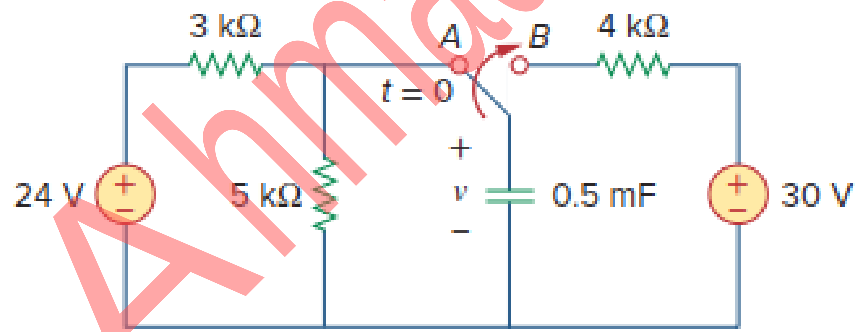
We obtain item 1 from the given circuit for $t < 0$ and items 2 and 3 from the circuit for $t > 0$

Note that if the switch changes position at time $t = t_0$ instead of at $t = 0$, there is a time delay in the response so that the **complete response** becomes:

$$v(t) = v(\infty) + \left[v(t_0) - v(\infty)\right]e^{-(t-t_0)/\tau}$$

where $v(t_0)$ is the *initial value* at $t = t_0^+$

Example 7.5. The switch in Fig. has been in position *A* for a long time. At $t = 0$, the switch moves to *B*. Determine $v(t)$ for $t > 0$ and calculate its value at $t = 1$ and 4 s.



Solution

For $t < 0$, the switch is at position ***A***. The capacitor acts like an open circuit to dc, but ***v*** is the same as the voltage across the 5-k Ω resistor.

Hence, the voltage across the capacitor just before $t = 0$ is obtained by voltage division as

$$v(0^-) = \frac{5}{5+3}(24) = 15\,\text{V}$$

The capacitor voltage cannot change instantaneously: $v(0) = v(0^-) = v(0^+) = 15\,\text{V}$

For $t > 0$, the switch is in position ***B***. The Thevenin resistance connected to the capacitor is R_{Th}= 4 kΩ, and the time constant is $\tau = R_{Th}C = 4\times10^3 \times 0.5\times10^{-3} = 2\,\text{s}$

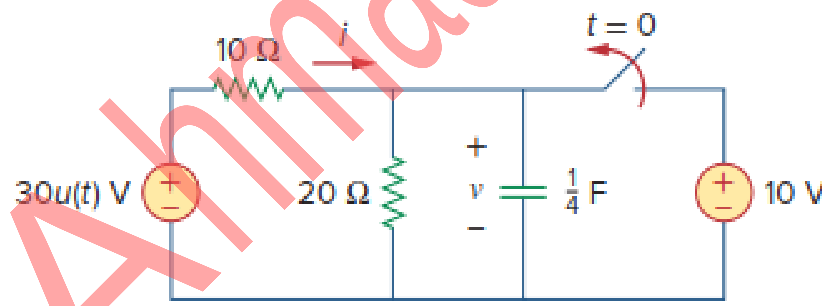
Since the *capacitor acts like an open circuit* to dc at steady state, $v(\infty)$ = 30 V. Thus,

$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau} = 30 + (15-30)\,\text{e}^{-t/2} = (30 - 15\,\text{e}^{-0.5t})\ \text{V}$$

$$t = 1 \Rightarrow v(1) = 30 - 15\,\text{e}^{-0.5} = 20.9\ \text{V}$$

$$t = 4 \Rightarrow v(4) = 30 - 15\,\text{e}^{-2} = 27.97\ \text{V}$$

Example 7.6. In Fig., the switch has been closed for a long time and is opened at $t = 0$. Find i and v for all time.



Solution

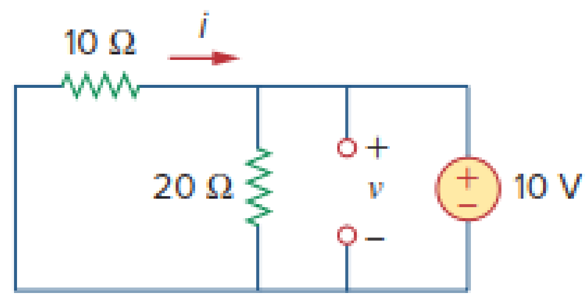
At $t = 0$, The current i can be discontinuous (منقطع), while the capacitor voltage v cannot.

Hence, it is always better to find v and then obtain i from v.

The unit step function is
$$30u(t) = \begin{cases} 0, & t < 0 \\ 30, & t > 0 \end{cases}$$

For $t < 0$,

1) the switch is closed and $30u(t) = 0$, so that the $30u(t)$ voltage source is replaced by a short circuit.
2) Since the switch has been closed for a long time, the capacitor voltage has reached steady state and the capacitor acts like an open circuit, Fig. (a).



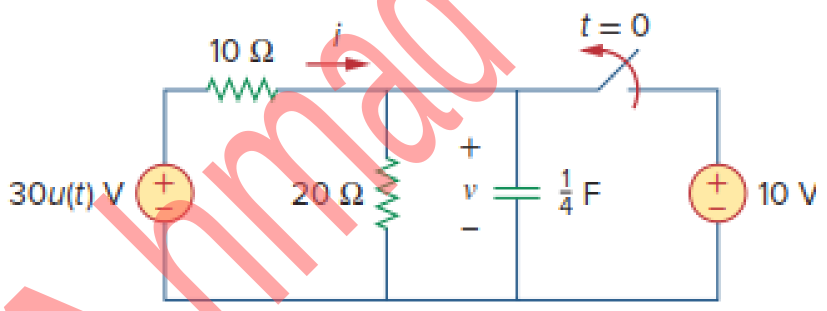
(a)

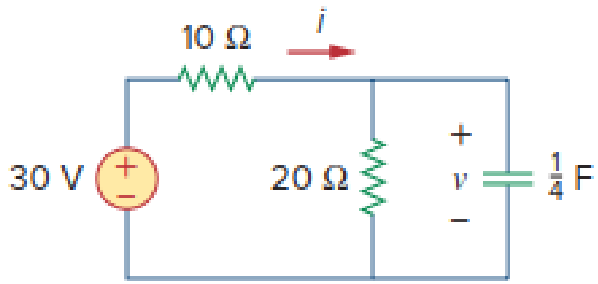
From this circuit we obtain
$$v = 10\,\text{V} \Rightarrow i = -\frac{v}{10} = -1\,\text{A}$$

Since the capacitor voltage cannot change instantaneously,
$$v(0) = v(0^-) = 10\,\text{V}$$

For $t > 0$,

1) the switch is opened and the 10-V voltage source is disconnected from the circuit.
2) The $30u(t)$ voltage source is now operative, Fig.(b).
3) After a long time, the circuit reaches *steady state* and the capacitor acts like an open circuit again.





(b)

We obtain $v(\infty)$ by using VDR, writing

$$v(\infty) = \frac{20}{20+10}(30) = 20\,\text{V}$$

The Thevenin resistance at the capacitor terminals is

$$R_{\text{Th}} = 10 \,||\, 20 = \frac{20}{3}\,\Omega$$

The time constant is

$$\tau = R_{\text{Th}}C = \frac{20}{3} \times \frac{1}{4} = \frac{5}{3}\,\text{s}$$

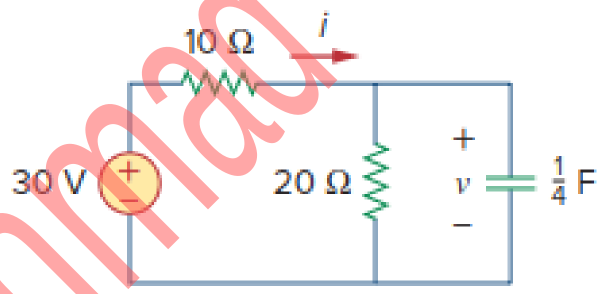
Thus,

$$v(t) = v(\infty) + \left[v(0) - v(\infty)\right]e^{-t/\tau} = 20 + (10-20)\,\text{e}^{-(3/5)t} = (20 - 10\,\text{e}^{-0.6t})\ \text{V}$$

To obtain i, from Fig. (b):

$$i = \frac{v}{20} + C\frac{dv}{dt} = \frac{20 - 10\mathrm{e}^{-0.6t}}{20} + \frac{1}{4}\cdot\frac{d(20 - 10\mathrm{e}^{-0.6t})}{dt}$$

$$= 1 - 0.5e^{-0.6t} + 0.25(-0.6)(-10)e^{-0.6t} = (1 + e^{-0.6t})\,\mathrm{A}$$



(b)

(Alternately, $v + 10i = 30\mathrm{V}$)

Finally,

$$v = \begin{cases} 10\,\mathrm{V}, & t < 0 \\ (20 - 10e^{-0.6t})\,\mathrm{V}, & t \ge 0 \end{cases}; \quad i = \begin{cases} -1\,\mathrm{A}, & t < 0 \\ (1 + e^{-0.6t})\,\mathrm{A}, & t \ge 0 \end{cases}$$

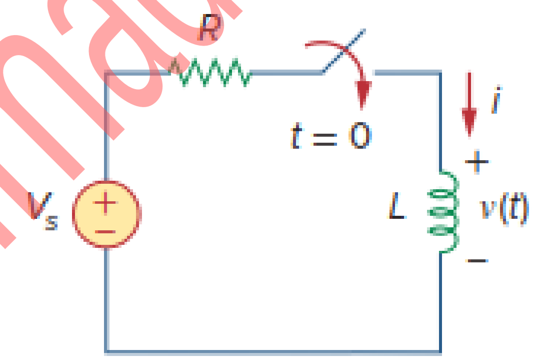
7.5 The Step-response of a RL Circuit

- The **step response** of a circuit is its behavior when the excitation is the step function, which may be a voltage or a current source.
- The circuit response is the inductor current ***i*** =???.
  - The *response* is the sum of the *transient response* and the steady-state response: $i = i_t + i_{ss}$
  - The *transient response* is always a decaying exponential: $i_t = Ae^{-t/\tau}, \quad \tau = \frac{L}{R}, \quad A -$ constant
  - The *steady-state response* is the value of the current a long time after the switch is closed.

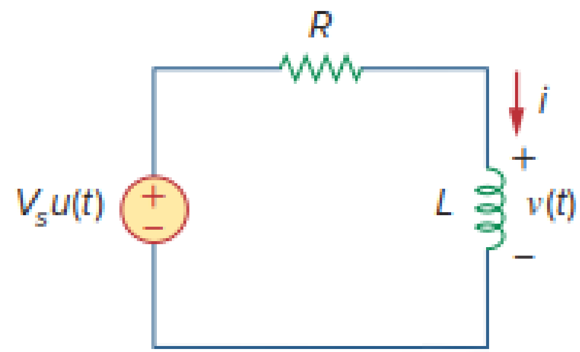
$$i_{ss} = \frac{V_s}{R}$$

  - Thus,

$$i = Ae^{-t/\tau} + \frac{V_s}{R}$$



(a)

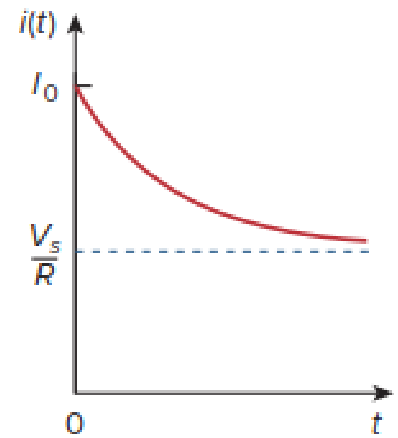


(b)

- Since the current through the inductor cannot change instantaneously,
  - at $t = 0$, the **initial current** is $i(0^+) = i(0^-) = I_0$

$$i = Ae^{-t/\tau} + \frac{V_s}{R} \Rightarrow I_0\Big|_{t=0} = A + \frac{V_s}{R} \Rightarrow A = I_0 - \frac{V_s}{R}$$

at $t = \infty$, the **Final inductor current** is $i(\infty) = \frac{V_s}{R}$



  - This is the **complete response** of the *RL* circuit

$$i(t) = \frac{V_s}{R} + \left(I_0 - \frac{V_s}{R}\right)e^{-t/\tau} \Leftrightarrow i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau}$$

Three steps to find out the step response of an *RL* circuit:

1. The initial inductor current $i(0)$ at $t = 0$.
2. The final inductor current $i(\infty)$.
3. The time constant τ.

- at time $t = t_0$: $$i(t) = i(\infty) + [i(t_0) - i(\infty)]e^{-(t-t_0)/\tau}$$

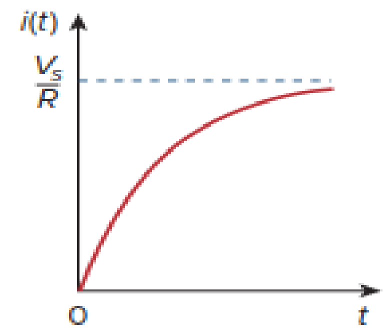
- If $I_0 = 0$, then, the *step response* of the *RL* circuit with no initial inductor current.

$$i(t) = \begin{cases} 0, & t < 0 \\ \dfrac{V_s}{R}(1 - e^{-t/\tau}), & t > 0 \end{cases} \Leftrightarrow i(t) = \frac{V_s}{R}(1 - e^{-t/\tau})u(t)$$

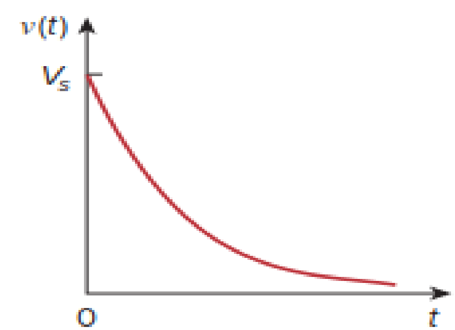
- The **voltage** across the inductor is

$$v(t) = L\frac{di}{dt} = L\frac{d}{dt}\left(\frac{V_s}{R}(1 - e^{-t/\tau})\right) = V_s \frac{1}{\tau}\frac{L}{R}e^{-t/\tau}$$

- With $\tau = \dfrac{L}{R}$ ➔ $$\boxed{v(t) = V_s e^{-t/\tau} u(t)}$$
-



(a)

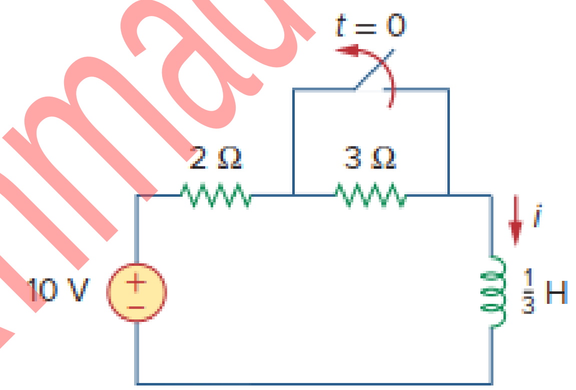


(b)

(a) current response.

(b) voltage response.

Example 7.7. Find $i(t)$ in the circuit of Fig. $t > 0$. Assume that the switch has been closed for a long time.



Solution

1) When $t < 0$, the 3-Ω resistor is short-circuited, and the inductor acts like a short circuit, so that

$$i(0^-)\Big|_{t=0^-} = \frac{10}{2} = 5\,\text{A}$$

Because the inductor current *cannot change instantaneously*, $i(0) = i(0^-) = i(0^+) = 5\,\text{A}$

2) When $t > 0$, the switch is open. The 2- and 3- Ω resistors are in series, so that

$$i(\infty) = \frac{10}{2+3} = 2\,\text{A}$$

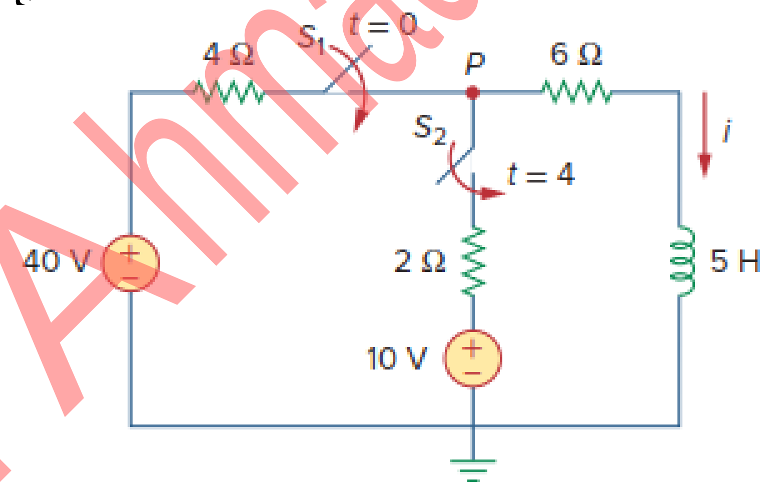
The Thevenin resistance across the inductor terminals is $R_{\text{Th}} = 2+3 = 5\Omega$

3) The time constant,

$$\tau = \frac{L}{R_{\text{Th}}} = \frac{\frac{1}{3}}{5} = \frac{1}{15}\ \text{s}$$

Finally, $i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau} = 2 + (5-2)e^{-15t} = 2 + 3e^{-15t}\ \text{A},\ t > 0$

Example 7.8. At $t = 0$, switch 1 in Fig. is closed, and switch 2 is closed 4 s later (time delay).
Find $i(t)$ for $t > 0$. Calculate i for $t = 2$ s and $t = 5$ s



Solution

We need to consider the three time intervals $t \le 0$, $0 \le t \le 4$, and $t \ge 4$ separately.

For $t < 0$, switches S1 and S2 are open so that $i = 0$:

$$i(0^-) = i(0) = i(0^+) = 0$$

For $0 \le t \le 4$, S1 is closed and S2 is still open, so that the 4- and 6- Ω resistors are in series.

Hence,

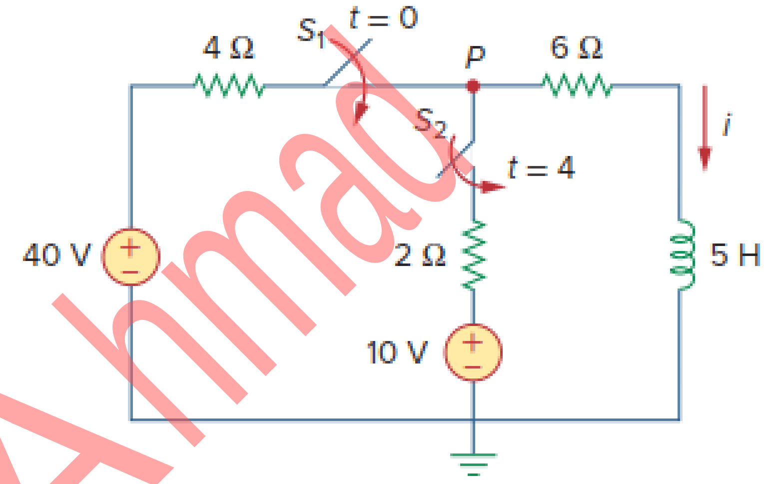
$$i(\infty) = \frac{40}{4+6} = 10\,A, \quad R_{Th} = 4+6 = 10\Omega, \quad \tau = \frac{L}{R_{Th}} = \frac{5}{10} = \frac{1}{5}\text{s}$$

Thus,

$$i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau} = 4 + (0-4)e^{-2t} = 4(1 - e^{-2t}) \text{ A}, \quad 0 \le t \le 4$$

For $t \geq 4$, S2 is closed (S1 is still closed); the 10-V voltage source is connected, and the circuit changes. This *sudden change* does not affect the inductor current because the current *cannot change abruptly* (مفاجئ). Thus, the initial current is

$$i(t)\big|_{t=4} = i(4) = i(4^-) = 4(1-e^{-8}) \simeq 4\,\text{A}$$



To find $i(\infty)$, let v be the voltage at node P in Fig. Using KCL,

$$\frac{40-v}{4} + \frac{10-v}{2} = \frac{v}{6} \Rightarrow v = \frac{180}{11}\,\text{V}, \qquad i(\infty) = \frac{V_s}{R} = \frac{v}{6} = 2.727\,\text{A}$$

The Thevenin resistance at the inductor terminals is

$$R_{\text{Th}} = (4\,\|\,2) + 6 = \frac{22}{3}\,\Omega \Rightarrow \tau = \frac{L}{R_{\text{Th}}} = \frac{5}{R_{\text{Th}}} = \frac{15}{22}\,\text{s}$$

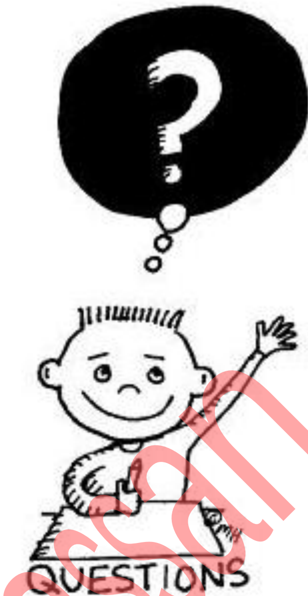
Hence,

$$i(t) = i(\infty) + [i(t_0) - i(\infty)]e^{-(t-t_0)/\tau}\Big|_{t_0=4} = 2.727 + [4-2.727]e^{-(t-4)/\tau}, \quad \tau = \frac{15}{22}$$

$$= 2.727 + 1.273e^{-1.4667(t-4)}, \quad t \geq 4$$

Finally,

$$i(t) = \begin{cases} 0, & t \leq 0 \\ 4(1-e^{-2t}), & 0 \leq t \leq 4 \\ 2.727 + 1.273e^{-1.4667(t-4)}, & t \geq 4 \end{cases}$$



The end of chapter 7